Gödel's Incompleteness Platonism exempts *Principia Mathematica*

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Abstract

Gödel's article "On Formally Undecidable Propositions of *Principia Mathematica* and related systems" (1931), offered in its title the promise of obtaining an important incompleteness result concerning Whitehead and Russell's *Principia*. I want to argue that, taken literally, it fails to make good on this promise. Of course, one may feel justified in interpreting the promise as having been made in the context, not of *Principia* itself, but of the modifications to *Principia* Gödel thought are needed to make it viable as a theory in which natural numbers are abstract particulars that are identified as classes under an ontology of simple types of classes. Fair enough. But it remains to evaluate Gödel's first theorem as applied to the actual *Principia* (modified only by adding its wff *infin ax* as a new axiom). If we take seriously *Principia* thesis that there are no natural numbers as abstract particulars and that classes are not simple types of entities, Gödel's first theorem cannot apply. Its famous diagonal function does not exist.

1 Background: Revolution *within* Mathematics

This paper endeavors to reveal that Gödel's important first incompleteness theorem does not hold in Whitehead and Russell's *Principia Mathematica* even when supplemented by an axiom assuring infinity. It does not hold because it is based on a thesis that natural numbers are abstract particulars. This has often been missed because it has been assumed that although *Principia* is officially eliminatist about the metaphysical ontology of natural numbers as abstract particulars and the recursive functions defined on them, it fully reconstructs arithmetic and the operations of *addition* and *multiplication*. I will argue that although it does indeed capture what it regards as arithmeitc, it does not capure what Gödel's Platonic ontology of numbers as abstract particulars countenances as arithmetic. Gödel's ontological Platonism allows recursive functions that do not exist by the lights of the revolution in mathematics against abstract particulars that Whitehead and Russell so lauded. In particular, Gödel's famous "diagonal" recursive function does not exist.

It has been long forgotten that the Whitehead-Russell Logicism of *Principia Mathematica* endorsed a revolution within mathematics against abstract particulars (numbers, geometric figures, sets/classes) in any of its branches. This revolution *within* is not itself a form of logicism. It is an independant movement. The Whitehead-Russell logicism fully embraced it. Frege's logicism rejected it, and so also did Zermelo and a good many others who rejected logicism itself. As early as 1901 and the paper "Mathematics and the Metaphysicians," Russell wrote:

> One of the chief triumphs of modern mathematics consists in having discovered what mathematics really is, ... All pure mathematics – Arithmetic, Analysis, and Geometry– is built by combinations of the primitive ideas of logic [i.e., the study of relational structures] (*MM*, p. 75)
>
> The solution of the problems of infinity has enabled Cantor to solve also the problems of continuity. ... The notion of continuity depends on that of order, since continuity is merely a particular type of order. Mathematics has, in modern times, bought order into greater and greater prominence. ... The investigation of different kinds of series and their relations is now a very large part of mathematics, and it has been found that this investigation can be conducted without any reference to quantity, and for the most part, without any reference to number. All types of series are capable of formal definition, and their properties can be deduced from the principles of symbolic logic by means of the Algebra of Relatives [i.e., the impredicative comprehension of relations and the study of relational structures]. ... nowadays the limit is defined... This improvement also is due to Cantor, and it is one which has revolutionized mathematics. Only order is not relevant to limits. ... Geometry, like Arithmetic as been subsumed, in recent times, under the general study of order (*MM*, p. 92).

In Russell's view, what mathematicians were studying all along was relations and the structure types that they determine can be studied indepentently of the contingencies of their exemplification. This revolution, which Russell regarded as largely inaugurated by the work of Cantor and his conception of number (cardinal and ordinal) in terms of 'similarity' (and in the case of ordinals, relation-similarity). *Principia* aimed to be its flagship, and it does so by maintaining that all of mathematics, as a study of relational structures, is captured as a synthetic *a priori* science which embodies the impredicative comprehension of properties and relations in intension. Impredcative comprehension is canonized in *Principia* by the axiom schemas *12.1.11 of volume 1, which render impredicative comprehension for properties and dyadic relations in intension.

Whitehead pointed out in a letter to Russell of 12, July 1910 that the fourth volume on geometry would embrace further impredicative comprehension axiom schemas for triadic and higher adicity relations in intension.[1]

Unfortnately, the revolution within mathematics was lost in the fog produced by the discovery of paradoxes with the naive notion of a class and by a quite different metaphysical movement that advocated the set-theoretical take over of mathematics. Gödel did not join the revolution in mathematics. He, like Zermelo, vonNeumann, and later Putnam, were relentless advocates of a set-theoretical take over, and of course sets are (under any conception) abstract particulars. Frege's Logicism was *not* part of the revolution within mathematics against abstract particulars. Contrary, to Boolos (1987, 1990) and neo-Logicists such as Wright (1983), his greatest achievement is not to have discovered that a theory of natural numbers (as abstract particulars) derives directly from Hume's Principle. His achievement, already present in his *Begriffsschrift* (1879), was to have discovered a conception of Logic according to which it embodies the impredicative comprehesion of functions in a hierarchy of levels. Impredicative comprehension of functions, he explained, reveals the logical foundation of the 'ancestal' relation which is the source of the distinctive kind of *induction* found in arithmetic proof. In fact, Frege's thesis that logic embodies impredicative comprehension forms a second revolution– a Fregean revolution in Logic which I call "cpLogic."

Frege held that functions are not objects, but are are unsaturated entities that thereby must come in a hierarchy of levels. All the same, Frege never doubted that numbers are objects. His mature *Grundgesetze* (1893) rejected classes/sets as part of arithmetic ontology, and maintained that cpLogic assures that there exists a special one-to-one heterogeneous function $\dot{z}\Phi z$ that takes first-level functions $f\xi$ to objects $\dot{z}fz$. Cardinal numbers are, according to Frege, objects knowable only as the objects that are the values of this heterogeneous function. Frege's correlation was designed to yield the theorem:

$$\vdash\ x \frown \dot{z}fz = fx.$$

Numbers are thereby logical objects which are correlates of second-level numeric functions. Thus, for instance, $0_x\Phi x$, i.e., $\overset{x}{\smile}\ \neg\Phi x$ is a second-level quantificationally numeric function correlated with the logical object 0, i.e., $\dot{z}0_x(x \frown z)$ which is a purely logical particular (*object* in his technical sense). Of course, the existence of Frege's second-level function turned out to be impossible because its converse function conflicts with a heterogeneous variant of Cantor's power theorem– his theorem which assures that there can be no function from objects onto functions.

Frege's logicism rejected the revolution within mathematics against abstract particulars in any of its branches. Whitehead-Russell Logicism of *Principia* accepts the revolution within mathematics against abstract particular and regards it as largely inaugurated by the work of Cantor. Now *Principia* also accepts

[1]See the Whitehead-Russell correspondance, Nicholas Griffin ed., forthcoming.

Frege's revolutionary cpLogic (though it makes it comprehend relations instead of functions). But as we can see, Frege's logicism is quite antithetical to the Whitehead-Russell logicism.

Both the revolution within mathematics against abstract particulars and Frege's revolutionary cpLogic, it should be noted, were entirely independent of any logicism. The anti-revolutionary metaphysicians advocating sets, of course, try to tell a different story of the history of logicism. They present the myth that there was a successful "reduction" of mathematics to set theory (though they never say to which set theory). They present logicism (whether Frege's or that of Whitehead-Russell) as endeavoring a *further* reduction project, which captures within logic that allegedly successful set-theoretic reduction. Zermelo's 1908 set-theory, one can unabashedly admit, enables the development of a good deal of mathematics. But that came well after logicism was well under way. In any case, the metaphysicians tell a history according to which logicism was the valiant but unsuccessful program of a reduction of (some such) set theory (or theory of classes) to logic. They never question whether the metaphysical ontology of some or another "correct" set theory (or theory of classes) is required for the branches of mathematics. Gödel viewed the Whitehead-Russell logicism in just that way, ignoring the revolution against abstract particulars which it so lauded. In his contribution "Russell's Mathematical Logic" to the Schilpp volume on Russell, he wrote:

> But in Russell the paradoxes has produced a pronounced tendency to build up logic as far as possible without the assumption of the objective existence of such entities as classes.... This led to the formulation of the aforementioned "no-classes theory" according to which classes ... were to be introduced as *façon de parler* (Gödel, 1944, p. 141).
>
> It seems to me that the assumption of such objects [as classes] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory of theory of our sense perceptions... (Gödel 1944, p. 137).

It is quite unfortunate that Russell never included a reply in the Schilpp volume to Gödel paper. It significantly misrepresents his work. Mathematics embraces fundamentally extensional contexts. Early on it was thought that classes and relations-in-extension (which are extensional entities) innocuously shadow the theory of properties and relations (which are intensional entities). Nothing comes by way of the former, it was naively thought, that is not already present in the latter. The relevant paradoxes (e.g., Russell's, Cantor's paradox of the greatest cardinal, Burali-Forti's paradox of the greatest ordinal) revealed that the "extensional shadow" notion of a class/set (and relation in extension) could no longer be regarded as an innocuous technique for dealing with extensionality. But this by no means wins the day for the metaphysicians imposing abstract particulars into the branches of mathematics. Russell found that by simple appeal to scope distinctions, one can recover extensional contexts from the intensional contexts of properties and relations as intensional entities. There is no need to employ extensional entities to recover the extensionality of mathematical contexts. Thus, once again, another metaphysician's indispensability argument for abstract particulars in mathematics fails. Whitehead and Russell's *Principia* embrace the revolution within mathematics against abstract particulars. The fog produced by the relevant paradoxes sadly focused attention away from the revolution.

In any event, it is simply misguided to think that Whitehead-Russell logicism endeavors give a further reduction to logic of the set theory (whatever the set theory may be) that gives the ontology of abstract particulars necessary for mathematics. There was never any such attempt in *Principia*, for the work never accepted that the metaphysicians of abstract particulars have legitimate authority to set the agenda for what the revolutionaries must recover in order to be successful. Nevertheless, to this day such metaphysicians take themselves to have such an authority. Obviously, this is question begging and no revolutionary mathematician should feel obligated to recover what (as Russell put it in his *A History of Western Philosophy, p. 829*) are "muddles" produced by the metaphysicians working under intutions of special kinds of necessity governing abstract particulars. There are no numbers as abstract particulars according to the revolutionary mathematicians railing against the metaphysicians. The question before us is the impact this has on Gödel's first incompleteness theorem as it pertains to *Principia*, modified only by adding its wff *infin ax* as a new axiom.

2 Definite Descriptions and Gödel's G

Gödel's first incompleteness theorem, i.e., what may better be called "Gödel's negation incompleteness theorem" (more carefully, the Gödel-Rosser negation incompleteness theorem), can be stated as follows:

> Every consistent and recursively axiomatic system K in which every recursive function is representable is negation incomplete– i.e., there is a wff G of the formal language of the system such that $\nvdash_K G$ and $\nvdash_K \sim G$.

Our task is to see whether we can get this result when the system K is *Principia Mathematica* (PM).

Now first and foremost, the representability of recursive functions is quite important to the viability of Gödel's incompleteness theorem. But the functions Gödel invisioned all take numbers as abstract particulars as their arguments and their values. This has to be avoided from the onset. In particular, where R is a dyadic relation between numbers as astract particulars, we need:

If Rmn then $\vdash_K\ \mathbb{R}(\bar{m},\bar{n})$

If not Rmn. then $\vdash_K \sim \mathbb{R}(\bar{m},\bar{n})$,

where $\mathbb{R}$ is a wff of the language of K that expresses the relation R between numbers m and n as abstract particulars, and $\overline{m}$ and $\overline{n}$ are numerals in the language of theory K for these numbers (respectively). Similarly, we have

If $fm = n$. then $\vdash_K\ (\forall y)(F\bar{m}y \equiv y = x)$,

where, for any dyadic functional relation f, a given finite recursive walk down to its base is represented by wff $\mathcal{F}$ in K. In particular, Gödel needs:

If m is the gln of a proof of the wff with gln n, then $\vdash_K\ Bew^K(\bar{m},\bar{n})$.

If m is not the gln of a proof of the wff with gln n, then

$\vdash_K \sim Bew^K(\bar{m},\bar{n})$.

I'm using $Bew^K(z,v)$ to abbreviate a wff representing "z is the Gödel number (gln) of a proof in system $K$of the wff with gln v."

Now *Principia's* language has no numerals for natural numbers. The appearance that 0 and 1 are numerals is misleading since such expressions vanish by application of the contextual definitions that eliminate class expressions. For instance, the expression Nc'$\hat{z}Pz = 0$ becomes,

$(\exists\Sigma)(\Sigma!\sigma\ \equiv_\alpha\ \sigma \approx \hat{z}Pz\ .\&\ .$
$(\exists\theta)(\theta!\alpha\ \equiv_\alpha\ (x)(x \notin \alpha)\ .\&.\ \Sigma! = \theta!))$.

There are no numerals for numbers at all, and there is no reductive identification of numbers with anything in *Principia's* ontology. One finds:

$0 = df_{*54.01}\iota\Lambda$

$1 = df_{*52.01}\hat{\alpha}(\exists x)(\alpha = \iota x)$

$\sigma \underset{c}{+} 1 = df_{*110.02}\ \hat{\xi}(\exists\alpha,\beta)(\sigma = N_oc\text{'}\alpha\ \&\ 1 = N_oc\text{'}\beta\ .\&.\ \xi\ sm\ \alpha + \beta)$

$(1\underset{c}{+})\text{'}0 = df_{*30,*38}\ (\iota\delta)(\delta = 1 \underset{c}{+} 0)$.

Thus, to capture the representability and expressibility that Gödel needs without assuming that natural numbers are abstract pariculars named by numerals of the langauge of K, let us use the notation:

$(1\underset{c}{+})^{m+1}\text{'}0 = df\ 1 \underset{c}{+} (1\underset{c}{+})^{m}\text{'}0$.

In this expression, m is a numeral, but may be removed by repeated application of the definition. Thus, where PM abbreviates the system of *Principia*, we can write:

If $(1\underset{c}{+})^{m}$‘0 is the gln of a proof of the wff with gln $(1\underset{c}{+})^{n}$‘0, then
$\vdash_{PM}\ Bew^{K}((1\underset{c}{+})^{m}‘0,\ (1\underset{c}{+})^{n}‘0)$

If $(1\underset{c}{+})^{m}$‘0 is not the gln of a proof of the wff with gln $(1\underset{c}{+})^{n}$‘0, then
$\vdash_{PM} \sim Bew^{K}((1\underset{c}{+})^{m}‘0,\ (1\underset{c}{+})^{n}‘0)$.

The above is made viable because we can eliminate the class expressions in the antecedent clause wholly independently of eliminating the class expressions in consequent clauses concerning theorems of PM. So far so good.

What, we may naturally ask, is wff G? This is a matter of no small importance. To see the issues in the context of *Principia*, let us use Russellian definite descriptions as quantifiers. For those not familiar with the syntax of using definite descriptions as quantifiers, let us recall the following definition from *Principia*:

$$*14.01 \quad [\iota x\varphi x][\psi(\iota x\varphi x)] \ = df\ (\exists x)((\forall y)(\varphi y \ \equiv\ y = x)\ \&\ \psi x).$$

This definition is designed so that the scope marker can be dropped under the convention of smallest scope. But in what follows it is better to use the following:

$$[\iota x\varphi x][\psi x] \ = df\ (\exists x)((\forall y)(\varphi y \ \equiv\ y = x)\ \&\ \psi x).$$

It will be useful as well to employ:

$$*14.02 \quad E!(\iota x\varphi x) \ = df\ (\exists x)(\forall y)(\varphi y \ \equiv\ y = x).$$

Thus, using Russsellian definite descriptions, we have:

$G \ = df\ (\iota v)(S\bar{r}v)[(\forall z) \sim Bew^{K}(z, v)]$
the gln of “$(\iota v)(Sxv)[(z) \sim Bew^{K}(z, v)]$” $= r$
the gln of “$(\iota v)(S\overline{r}v)[(z) \sim Bew^{K}(z, v)]$” $= g$.

Here we are imagining, for the moment, that S stands in for a wff that represents the allegedly existing diagonal recursive function \$ which holds when n is the gln of the wff resulting from substituting the numeral $\overline{m}$ for the free variable “x” in the wff with gln m. What then does G say? If we were to read G literally, imagining the intended domain to consist solely of natural numbers as abstract particulars, we might say:

> There is a unique gln y of the wff obtained by taking the gln r of the wff , “$(\iota v)(Sxy)[(z) \sim Bew^{K}(z, v)]$” and substituting its numeral $\bar{r}$ in the position of its free variable “x”, and no natural number z is the gln of a proof in system K of the wff with gln v.

Where the gln of the wff “G” $= g$, and the diagonal function S is representable, we get:

$$\vdash_{K}\ (\forall y)(S\bar{r}y \ \equiv\ y = \overline{g}).$$

The key to Gödel's proof is to arrive at the following remarkable diagonal theorem:

$$\vdash_K \; G \;\equiv\; (\forall z) \sim Bew^K(z, \bar{g})].$$

If we write the diagonal theorem with the wff that is G, we get the following:

$$\vdash_K \; (\iota y)(S\bar{r}y)[(\forall z) \sim Bew^K(z, y)] \equiv \; (\forall z) \sim Bew^K(z, \bar{g})].$$

But we must next fact up to the elimination of numerals for the Gödel numbers r and g in the diagonal temma. To remove the numeral "$\overline{g}$" we can use the definite description:

$$(\iota y)(S\bar{r}y).$$

Thus, we get:

$$(Diag)^{PM} \qquad \vdash_K \; (\iota y)(S\bar{r}y)[\, G \equiv \; (\forall z) \sim Bew^{PM}(z, y)].$$

$$\vdash_{PM} \; (\iota y)(S\bar{r}y)[\quad (\iota v)(S\bar{r}v)[(\forall z) \sim Bew^{PM}(z, v)] \equiv (\forall z) \sim Bew^{PM}(z, y)].$$

This is quite illuminating. Note the importance of the primary scope of the definite description. A secondary scope would make this result the utterly trivial:

$$\vdash_{PM} \quad [(\iota v)(S\bar{r}v)[(\forall z) \sim Bew^{PM}(z, v)] \equiv$$
$$[(\iota y)(S\bar{r}y)][(\forall z) \sim Bew^{PM}(z, y)].$$ [2]

Thus, the primary scope is central, and we see that what appears as if it were a biconditional diagonal theorem, is now revealed to not be a biconditional at all; it is an existential wff assuring the unique existence of the value of an allegedly existing diagonal function \$. That is, we have a primary scope of the definite description $(\iota y)(S\bar{r}y)$ of a number that is the value of a specific case of the \$ function for the number r as argument. The diagonal temma relies on there being such a function as \$ and its unique outcome for a specific number as argument. But what is this number as argument? We cannot assume, without begging questions against the revolution within mathematics that there are functions (recursive or otherwise) from natural numbers as abstract particulars to natural number that are abstract particulars. We need to be able to get the following:

$$\vdash_{PM} E!(\iota y)(S\bar{r}y).$$

So far so good. This replaces the reliance on the numeral $\overline{g}$ in

$$\vdash_K \; (\forall y)(S\bar{r}y \;\equiv\; y = \overline{g}).$$

Now we can say the following holds:

$$[(\iota y)(S\bar{r}y)][\text{ the gln of "}(\iota y)(S\bar{r}v)[(z) \sim Bew^{PM}(z, v)]\text{"} = y].$$

[2] This would not be trivial at all if "S" were a free function variable rather than standing in as it does for a closed wff, but neither then would it be a theorem.

But that is not yet enough. We still have to eliminate the reliance on numeral $\overline{r}$. What to do?

We are left with a serious worry. How can we replace the numeral "$\overline{r}$" while staying within the object-language of *Principia*? If we take the same approach that was successful for avoiding a numeral "$\overline{g}$," we can try the following definite description:

$$(\iota w)(\text{the gln of "}(\iota v)(Sxv)[(z) \sim Bew^{PM}(z,v)]\text{"} = w).$$

Let's abbreviate this with "$(\iota w)(\Upsilon w)$". Now our Diagonal Lemma is this:

$$*(Diag)^{PM} \qquad \vdash_{PM} [(\iota w)(\Upsilon w)][\,(\iota y)(Swy)[G \equiv (\forall z) \sim Bew^{PM}(z,y)]].$$

$$i.e., \vdash_{PM} [(\iota w)(\Upsilon w)][\,(\iota y)(Swy)[(\iota v)(Swv)[(\forall z) \sim Bew^{PM}(z,v)] \equiv (\forall z) \sim Bew^{PM}(z,y)]]$$

We now see that our worry has become an insurmountable problem. The definite description "$(\iota w)(\Upsilon w)$" is not in the object-language of *Principia* because Υw is not in its object language. And there is no replacement. We cannot eliminate the numeral "$\overline{r}$". The discovery that we cannot eliminate the numeral "$\overline{r}$" focuses attention on the question as to whether, by the lights of *Principia*, there is a (recursive) function \$ in the first place!

In fact, if there are no natural numbers as abstract particulars, then there is every reason to doubt that there exists Gödel's diagonal recursive function \$. There is an unintended misdirection going on that obscures this. One imagines that there must be a recursive diagonal function \$ since it seems to be just one among the functions which, the metaphysician of numbers imagines *Principia's* Logicism to be under oath to emulate. Indeed, when seen from the eye of the metaphysician of numbers as abstract particulars, it appears as though Gödel's diagonal function lives within the functions of *addition* and *multiplication* themselves. If these are emulated, isn't it the case that one also emulates the existence of the diagonal function \$? No. This is misguided. The assumption of the existence of the function \$ begs the question against the revolution within mathematics against natural numbers as abstract particulars. There cannot be a function \$ unless natural numbers are abstract particulars.

Speaking on behalf of the perspective, not of Whitehead-Russell Logicism, but of the revolution within mathematics, there is a substantive question as to whether there is such a function as \$. The status of such a function in *Principia*, after all, simply reflects the revolution. Hence the issue does not turn on Whitehead-Russell logicism, but on the revolution itself. The G sentence is much about the function \$, as it is about natural numbers. For it speaks descriptively of the unique the value of function \$ as represented by S. This is issue is hidden by the usual narrative explication of Gödel's theorem which never doubts that there is a diagonal function \$ but only raises and positively answers the question as to whether such a function is recursive. And once it demonstrates that the function \$ recursive, then one knows it to be representable. After all, Gödel proved that every recursive function on natural numbers is representable in every adequate (consistent, axiomatizable) theory of elementary arithmetic. But that characterization leaves out a key element– namely, that Gödel was presuming that recursive functions take argument and have values that are natural numbers as *as abstract particulars*. Gödel's negation incompleteness theorem requires natural numbers to be abstract particulars.

We now see how one could be justified in questioning the very existence of a diagonal function \$. That is the key. How is it that the existence of a diagonal function \$ is questionable and yet functional *relations* such as *multiplication* and *addition* (also characteristic functions and the like) are not similarly questionable? The answer, according to the revolution within mathematics, is that it is question-begging to assume that *multiplication* and *addition* are functions on natural numbers as abstract particulars. That is, it begs the question against the revolution. Gödel was no revolutionary. His Platonism vehemently rejected it, and his diagonal function \$ essentially *does* require a metaphysics of numbers as abstract particulars.

Now there is no problem with Gödel numbering in general. *Principia* can emulate there being such 1-1 relations, and in virtue of a given relation, a sign ")" can be uniquely correlated to, say, the number 5. But this is not a relation between individuals of type o and individuals of type ((o)). *Principia's* account expresses numeric relations without making numbers individuals of any type. There are no numbers. Indeed, appearances to the contrary, we have seen that there are no numerals whatsoever in the formal language of *Principia*. Of course, one could consistently add demumerably many individual constants (for every simple type– given are adding an axiom of infinity too. But still nothing would bring about an emulation of Gödel's function \$. Russell accepted the revolution within mathematics. Gödel rejected it—as does anyone today who reductively indentifies numbers with abstract particulars such as sets. It is that simple. Taking *Principia* seriously, how can one maintain that Gödel's recursive diagonal (substitution) function exists? I fear that one cannot. There is no Gödel's wff G in the language of *Principia*.

The issue I'm raising here is not about whether an ontology of simple types is untoward. The rejection of natural numbers as abstract particular is wholly independent of considerations fo simple types. Nor is the issue about infinity. The issue is that there are no numbers according to *Principia*, even interpreted as a Realism about simple types of universals. It follows that there is no reason to believe that Gödel's diagonal function \$ exists. Why? Because his diagonal function \$ exists only if natural numbers are abstract particulars! The same cannot be said for functions of *addition* and *multiplication* and so on for arithmetic. That is because the revolution within mathematics maintains that these relations are not committed to numbers as abstract particular. Now if *Principia* is the flagship of the revolution within mathematics against abstract particulars, how is it that Gödel got away with saying (as is embedded in the very title of his paper) that his results apply to *Principia*? The answer is clear enough: He didn't take *Principia's* seriously.. He assumed that the revolution within mathematics failed. He, therefore, has in mind an altered *Principia* which would embrace an ontological commitment to an infinity of abstract particulars that are numbers.

This is an important point that has been missed, and it is one which (as we shall see) might go some significant way toward better understanding the orientation of Russell (and Wittgenstein too) in reacting to Gödel. We shouldn't think it obvious that G is a statement about natural numbers as abstract particulars! Questions of *aboutness* are always philosophically slippery. To be sure, the quantifiers of K range, in the intended model Gödel imagined, over natural number as abstract particulars. So in *that* rather mundane sense, the G wff is about natural numbers. But in another very important sense, the G wff is every bit as much purports to be about a very distinctive diagonal recursive function which Gödel assumes, without argument, that *Principia* is *supposed to* emulate. That sense of *aboutness* we have a good philosophical reason to doubt that the wff G is appropriately about arithmetic. Gödel assumes the existence of a function \$ and in virtue of it, both $(Diag)^{PM}$ and G speak about one of its values. Thus, both say (assume without argument) something quite important–namely, that there exists of a recursive function \$ with the value in question. They presumes that there is a function \$. This simply begs the question against *Principia* which embraces the revolution in mathematics which rejects abstract particulars.

3 Foreground: Whitehead-Russell Logicism

The metaphysicians of abstract particulars have, according to the revolutionaries, no legitimate authority to set the agenda for what the revolutionaries must emulate. The revolutionary mathematicians are *not* required to to emulate whatever the metaphysicians of mathematics do with their numbers as abstract particulars. They have no legitimate authority to proclaim what mathematics is about, what mathematicians have been studying, what the revolutionary mathematics ought to emulate. As we have seen, Russell put it in his 1901 "Mathematics and the Metaphysicians," that the new mathematicians have finally discovered what their field is all about. This is of no small importance. It reveals that *Principia* is loyal to the revolution, not to repeating result wrought from intuitions of the metaphysicians imposing a metaphysics of abstract particulars upon mathematics. Mathematics studies all the kinds of structures there are by studying the way relations, independently of contingencies of their exemplification, order their fields. *Principia's* Logicism is, in this respect, is embracing the revolution within mathematics, not imposing it upon the field. Gödel was against the revolution in mathematics. Frege was against the revolution within mathematics. So was Zermelo who imposed his intuitions of Z-sets into its branches. So also against the revolution are Putnam (who would have one think that abstract mathematical particulars get justified by pragmatic considerations involving their alleged importance to empirical techniques of measurement in Physics) and Quine (who would have us believe that "ontological" questions about numbers are on a par with empirical contingent questions (e.g., about cats) and are wholly *relative*– there being no "absolute" facts about what there is). Whitehead and Russell embrace the revolution within mathematics. And Wittgenstein, in his own way alike offered their own philosophical doubts that numbers are abstract particulars, but the impetus (as Russell pointed out) comes from within mathematics itself which makes relational *order* the subject of mathematics. The implication of this is that the metaphysicans of abstract particulars are not properly in a position to set the agenda for what the revolutionaries have to emulate or recover. (No non-Euclidean geometer, for example, imagines she has to recover the metaphysician's claim that the Pythagorean theorem is a geometric necessity holding of abstract particulars that are right triangles.) In light of the revolution, it is the metaphysicians of abstract particulars that are the outsiders– imposing upon what the revolutionaries take to be the actual practices conducted within the fields of mathematics by mathematicians who are studying relations (relational structures). It is the metaphysicans of abstract particulars that have imposed abstract particulars upon the field. This is important for Whitehead and Russell's logicism, for it hopes to be the flagship of the revolution within mathematics. It hopes to explain the foundations of the various fields of the mathematical study of relations. Again, the burden, therefore, is not on Whitehead and Russell's Logicism to emulate what the metaphysicians of mathematics do with their abstract particulars.

Since the revolution within mathematics maintains that relational order is what mathematicians were studying all along, recursively characterized functions (many-one relations) which metaphysicians of mathematics, not the mathematicians themselves, misguidedly regard as committed to natural numbers as abstract particulars, are not regarded as being committed to abstract particulars at all. The burden is shifted. It is the metaphysicians with their numbers as abstract particulars that have the burden of finding an argument that mathematics depends upon their philosophical ontology (classes/sets, triangles or what have you), not the revolutionaries within mathematics and not the *Principia* Logicists who are simply representing what mathematicians do, in fact, study– namely relations. From this revolutionary perspective, we must be prepared to abandon some long cheristed results of the metaphysicians who imagined metaphysical necessities holding of their abstract particulars . The non-Euclidean geometers abandoned a good many– including the Parallel Postulate. But so also does *Principia* point out a good many others.

As noted earlier, *Principia* hoped to be the flagship of the revolution within mathmatics against the metaphysicians who conjured indispensability arguments for abstract particulars unique to its branches. Interesting, *Principia* also accepted (viewing a function as a many-one relation) the Fregean revolution in logic according to which pure logic impredicatively assures the existence of functions. *Principia* accepts axiom schema of impredicative (simple type) comprehension, namely

$*12.1 \quad (\exists f)(\varphi x \equiv_x f!x)$

$*12.11 \quad (\exists f)(\varphi xy \equiv_{x,y} f!xy)$, etc.

More generally stated, the clear intent is to have:

$$*12.n \quad (\exists f)(x_1, ..., x_n)(\varphi(x_1, ..., x_n) \equiv f!(x_1, ..., x_n)),$$

where $f!$ is not free in the wff φ. Recall that φx and φxy etc., are schematic for wffs and that $f!$ is for a genuine predicate variable. If were were to restore type indices, we can capture impredicative simple type comprehension with the following schema:

$$*12.n \quad (\exists f^{(t_1,...,t_n)})(x_1^{t_1}, ..., x_n^{t_n})(\varphi(x_1^{t_1}, ..., x_n^{t_n}) \equiv f^{(t_1,...,t_n)}(x_1^{t_1}, ..., x_n^{t_n})),$$

where $f^{(t_1,...,t_n)}$is not free in φ. Alas, almost all of this historical background as been lost (or forgotten). When we are reminded of it, we are reminded that there is much work to be done to make it well-known. I think the revolution within mathematics is alive and well with mathematicians today– most of whom are not at all interested in philosophical questions of ontology and its conundrums. But these sentiments have largely gone underground. The truth has sometimes been obscured by the well known fact that Russell tried, and tried, and tried again, to emulate impredicative (simple type) comprehension wiithout an ontology of simple types, while Frege embraced his levels of functions outright (and quite independently of any considerations of Russell's paradox). But taking *Principia's* uniquely non-Fregean logicism seriously, we should magine a fully Platonic Realist semantic interpretation of *Principia's* object-language predicate variables, i.e., its "individual" variables adorned with simple type indices such as x^o, $x^{(o)}$, $x^{((o))}$,etc., and $x^{(o,o)}$, $x^{(o,(o))}$, etc for relations– homogeneous or otherwise. (For convenience, one may use $\varphi^{(o)}$,$\varphi^{((o))}$,$\varphi^{(o,o)}$,$\varphi^{(o,(o))}$ and so forth, for the "individual" variables that are predicate variables– i.e., those whose simple type index is not o.) The natural interpretation of the syntax would adopt an ontology of universals (properties and relations in intension) regimented into simple types. Taking *Principia* seriously entails that it works without an ontology of numbers, classes/sets or other abstract particulars in mathematics. Now who has the burden of argument? Is it *Principia* that has the burden of emulating what the metaphysicians of numbers as abstract particulars do? Is its task to emulate an ontology of functions (i.e., functional relations) on natural numbers? Why let the metaphysician determine has to be done? That would begging questions in favor of the metaphysicians of mathematics who, like Gödel himself, never doubted that natural numbers are abstract particulars. Again, according to the revolution within mathematics, the mathematicians were never studying abstract particulars (numbers, geometric figures, points, etc) in the first place. They were studying relations.

One cannot, without question begging, hold *Principia's* Logicism hostage to the metaphysicians claims concerning special arithmetic necessities governed by their intuitions of numbers as abstract particulars. According to Whitehead and Russell, the revolution in mathematics has some startling implications. One that is rather strikingly unacknowledged, is the falsity of Hume's

Principle– which says that identity of cardinal numbers of classes assures that the classes are similar (equinumerous). On the basis of its non-homogeneous relations, *Principia* *100.321 shows that Cantor's power class theorem entails that Hume's Principle has exceptions when *descending* cardinals (those based on non-homogeneous similarity relations sm) are involved. This is so surprising that it is worth quoting Whitehead (*PM*, vol. 2, p. 15):

> $*100.321 \quad \vdash \alpha \text{ sm } \beta \,.\supset.\, Nc'\alpha = Nc'\beta$
>
> ... Note that $Nc'\alpha = Nc'\beta \,.\supset.\, \alpha \text{ sm } \beta$ is not always true. ... If $Nc'\alpha$, $Nc'\beta$ are descending cardinals, we may have $Nc'\alpha = \Lambda = Nc'\beta$ without having $\alpha \text{ sm } \beta$.

Of course, the reason we may have $Nc'\alpha = \Lambda = Nc'\beta$ in the descending cases the non-homogeneous relation of sm, is Cantor's power-class theorem. The point is that some of the cherished theses that the metaphysician of numbers as abstract particulars had thought to be (arithmetically) necessary are discovered by *Principia* to be not only not necessary, but not even true. The case is analogous to what happened in non-Euclidean geometry, where the metaphysicians were propounding special Euclidean "necessities" governing abstract particulars that are figures in space. Obviously, the revolutionaries think the metaphysicians of abstract particulars ought not to be in charge of the agenda of what they must recover.

Now a much vaunted case against *Principia's* approach, and therefore the approach of the revolutionaries within mathematics, is that it cannot recover the metaphysician's insistence that there are infinitely many numbers as abstract particulars. But why should it? The intuition of there being infinitely many natural numbers derives essentially from the assumption that numbers are abstract particulars and that "adding 1" necessarily produces more– something Cantor's infinitary cardinal arithmetic (e.g., $\aleph_0 + 1 = \aleph_0$) conclusively refuted.[3] Infinity, according to the revolution within mathematics, is not an arithmetically necessity. So the first codicil on Gödel's negation incompleteness theorem is that it needs to add *Principia's* Infinax as an antecedent clause (i.e., an antecedent clause assuring the infinity of its inductive cardinals). But let's not fret over that. There something very much more challenging we are worrying about that, as we have seen, holds even if we add the wff infin ax as a genuine axiom to *Principia.* According to the revolution in mathematics, inductive cardinals are not abstract particulars. Even with our Realist semantics for *Principia*, we cannot say that each inductive cardinal is to be identified with a simple typed universal. It is with the background of *Principia*+ *infin ax* that we negatively evaluated Gödel's first incompleteness theorem.

[3] See Landini (2011).

4 Russell and Wittgenstein: What does G say?

As we have see, one can reasonably philosophically doubt whether the there is a function $ at all. This has an important implications for the remarks given by Russell to Gödel's first theorem. Indeed, it has implications for Wittgenstein's remarks as well. I haven't uncovered new evidence that *this* concern about the existence of the diagonal function $ was precisely what animated Wittgenstein cryptic comments in the 1930's targeting Gödel's first theorem. But I suspect that such thoughts would quite naturally occur to him. Tractarian approach to arithmetic, as I see it, flatly rejected an ontology of numbers as abstract particulars. .[4] Wittgenstein hoped that that the entirety of arithmetic consists of combinatorial calculation of equations concerning recursively defined functions. (See *TLP*, 6.02, 6.021). Such recursive functions do not, according to Wittgenstein, require ontological commitments to natural numbers as abstract particulars. The *Tractatus* maintains that there are no numbers (as abstract particulars) and thus his equations for his arithmetic of operations characterized with *numeral* exponents do not embrace the Gödelian Platonic Realism needed for the existence of recursive function such as $ which, by its very nature, requires that numbers be abstract particulars.

What does G say? Of course, it does not say anything on its own. Obviously, one requires a Tarski-style formal semantic interpretation of the expression over a domain. Naturally, Gödel imagines the domain to be the abstract particulars he takes to be the natural numbers themselves– though since the system K is first-order Peano arithmetic we shall be forced to admit that there are non-standard models. Let's then answer the question, for the present, in the context of the presumption of a Tarski-style formal interpretation over a domain of natural numbers as abstract particulars, i.e., where we have consistency of K and the intended model N of K. Even in this context, we do not know whether G is true since we don't know whether the system K consistent. This has been pointed out many times. All the same, it is often said that we *do* know the following:

(*) If K is consistent, then G is true.

If we are careful, however, we shouldn't accept even this. What G says depends on the model of system K. Intuitively *truth* does not depend on any interpretation of any formal system. Now as already noted, Tarski's formal semantic definition requires "true-in-L," for formal langauge L of first-order system K. Gödel might well accept (with Tarski) that "is true," if it is to be made formal, must be "is-true-L", for some appropriate formal language L for system K. Therefore, what we know is this:

(**) If K is consistent (and so has a model m), then G is true-in-L in model m.

[4] See Landini (2020), forthcoming where Wittgenstein's approach to arithmetic is compared to the combinatorial approach of Fitch (1974).

We cannot know that m is the intended model, however.[5]

Several papers have taken up the question of whether Wittgenstein's remarks on Gödel's first incompleteness theorem are couched in ignorance of the logic and the semantics of the proof.[6] What, if anything well-informed, did Wittgenstein mean when he insinuated that the interpretation 'G is not provable'" of the Gödel formula G has to be given up if one assumes that either G or $\sim G$ is provable? More exactly, he said (§8 *Remarks on the Foundations of Mathematics, Appendix III, p. 118*):

> Just as we ask "'provable in what system?", so we must also ask "'true' in what system?" 'True in Russell's system' means, as we said: provable in Russell's system; and 'false in Russell's system' means the opposite has been proved in Russell's system. ... If you assume the proposition is provable in Russell's system, that means it is true *in the Russell sense*, and that interpretation "P is not provable" again has to be given up. If you assume that the proposition is true in the Russell sense, *the same* thing follows. Further: if the proposition is supposed to be false in some other than the Russell sense, then it does not contradict this for it to be proved in Russell's system. (What is called "losing" in chess may constitute winning in another game.)

Lampert has convincingly argued that, to date, attempts at seeing this comment as well-informed are not satisfactory and certainly far from indicative of a "remarkable insight".[7] Indeed, Lampert seems correct that Wittgenstein thought that in the derivation to arrive at $\nvdash_{PM} G$ and $\nvdash_{PM} \sim G$ one must reason using an interpretation of what G says. Wittgenstein's objection therefore misses the mark because the formal derivation does not require any interpretation of what G says. . Indeed, G says something only with respect to a model. But derivations do not depend on the interpretation of G in a model. Thus, Wittgenstein fails to diagnose a flaw in the derivation. Let's swap out Wittgenstein's P for our G. Clearly, Lampert is right that we cannot accept Wittgenstein's assumption that "G is true" means that G is provable in Russell's system; and "G is false" means that $\sim G$ is provable in Russell's system.

All the same, the last sentence of Wittgenstein's remark might provoke some interest. We might charitably interpret Wittgenstein as saying that "G is true" means that G is true-in-L in model m of Russell's system. If G is false in *this* other game (i.e., if G is false-in-L in the model m^* of PM), it certainly doesn't mean it is false in the natural numbers, for neither model m nor m^* need be N. And we cant distinguish any of these models by the lights of PM. Now G only says something with respect to our assigning it an interpetation in a model m, and we can only mean by "true-in-L in the model m" the fact that m is a

[5]Since *Principia* is second-order, all models of the natural numbers are isomorphic.

[6]See, for example, Floyd & Putnam (2000), Rodych (1999), Steiner (2001).

[7]See Lampert (2018).

model of PM. Hence, if were were to find that G were false in model $m*$ of PM, we should have to simply conclude, after all, that G is not provable in PM. The point Wittgenstein is after is then just that (*) is unwarranted, and all one legitimately can get is the unimportant result (**). This only seems to be important because of the pretense that "If PM is consistent, then G is true" assures the following:

> If PM is consistent, then G (an arithmetic wffs about natural numbers) true in N (the natural numbers).

Wittgenstein wants us to give up that pretense. Fair enough. But as Lampert points out, this would in no way undermine the Gödel formal derivation of negation incompleteness.

With Wittgenstein's remarks dispatched, naturally, one wonders whether Russell's remarks on Gödel's incompleteness theorems fare any better. On the surface, it seems not. Russell wrote (MPD, p. 114):

> In my introduction to the Tractatus, I suggested that, although in any given language there are things which that language cannot express, it is yet always possible to construct a language of higher order in which these things can be said. There will, in the new language still be things which it cannot say, but which can be said in the next language and so on ad infinitum. This suggestion, which was then new, has now become an accepted commonplace of logic. It disposes of Wittgenstein's mysticism and, I think, also of the newer puzzles presented by Gödel.

This passage should be read in conjunction with a more detailed passage in Russell's 1950 paper "Logical Positivism," p. 371, where we find:

> There has been a vast technical development of logic, logical syntax, and semantics. In this subject, Carnap has done the most work. Tarksi's *Der Begriff der Wahrheit in den formalisierten Sprachen* is a very important book, and if compared with attempts of philosophers in the past to define "truth" it shows the increase of power derived from a wholly modern technique. Not that difficulties are at an end. A new set of puzzles has resulted from the work of Gödel, especially in his article *Über formal unentscheidbare Säte der Principia Mathematica und verwqndter Systeme* (1931), in which he proved that in any formal system it is possible to construct sentences of which the truth or falsehood cannot be decided within the system. Here again we are faced with the essential necessity of a hierarchy, extending upwards ad infinitum, and logically incapable of completion.

Russell's claim that a hierarchy of senses of "truth" evades the "puzzle" is the same in both quotes from his writings on the subject. Later still, when he was 91, he wrote to Henkin confessing the following:

I realized, of course, that Gödel's work is of fundamental importance, but I was puzzled by it. It made me glad that I was no longer working at mathematical logic. If a given set of axiom leads to a contradiction, it is clear that at least one of the axioms must be false. ... You note that we [Whitehead and Russell] were indifferent to attempts to prove that our axioms could not lead to contradictions. In this, Gödel showed that we had been mistaken. But I thought that it must be impossible to prove that any given set of axioms does not lead to contradiction, and, for that reason, I paid little attention to Hilbert's work.

In the second part of the above, Russell is accepting (on independent intuitive grounds) Gödel's second incompleteness theorem which reveals the impossibility of Hilbert's program. But what is of interest is what Russell thought "puzzling" about Gödel's first incompleteness theorem. It will be noted that Russell does not say in this passage that Gödel's work is an outright diagonal paradox. But it seems that he did regard it as akin to such, since it is a "puzzle" that wants resolving.

As Russell well knew, there were many confusions about what wffs characterize genuinely mathematical recursive functions. He knew what we nowadays widely agree upon– namely, that confused appeals to wffs using "names" and "denotes" equivocally generate pseudo-paradoxes (such as the Richard, the König-Dixon, the Berry and later the Grelling) all of which have no import for mathematics. They are simply equivocations. It is important to remember that in 1906 Russell was well aware of this. He , dismissed the Richard, the König-Dixon, the Berry as a pseudo-paradox, writing (*STCR*, p, 185):

It seems to be defined as 'the class of definable ordinals'; but definable is relative to soem given set of fundamental notions, and if we call this set of fundamental ntions I, 'definable in terms of I' is never itself definable in terms of I. ... It is easy to define 'definable in terms of I' by means of a larger apparatus I*; but then 'definable in term sof I*' will require a still larger apparatus I** for its definition, and so on.

Russell never put "truth" into the category of "names," and "defines" because he thought that unlike the latter its doesn't, by its nature, require connection to a language of fixed signs. (This is quite compatible with the fact that Tarski's notion of "truth-in-L" requires fixed formal language L because the formal interpretation of a language over a domain requires fixing the terms and wffs of the language and also fixing (independently) the members of the domain of interpretation.) Russell noted in his discussion of Gödel that in his 1922 introduction to Wittgenstein's *Tractatus* he had suggested a hierarchy of notions of "truth" dispatches any would-be conundrum arising from Gödel's result just as assuredly as it did against Wittgenstein's Tractarian ouroboric mysticism according to which arithmetic and logic must be self-completing.

I don't mean to say that Russell confusedly regarded Gödel's first incompleteness theorem as one among such semantic pseudo-paradoxes as the Richard. That is not the point. The point is that his intuition may well have been that Gödel's diagonal construction ought to raise *suspicions* and red flags concerning which purported diagonal constructions are, in fact, genuine. This is the tie in with our earlier discussion which questions the very existence of Gödel's diagonal function $. If one accepts that recursive functions in mathematics are not genuinely such that their arguments and values are of natural numbers as abstract particulars, then one might well have very good philosophical grounds for regarding Gödel's diagonal function $ as every bit as non-existent as the Barber that shaves all and only those who do not shave themselves. The G sentence purports to be about an allegedly diagonal recursive function $. There is no such function. In light of this concern, Russell comments on Gödel's first incompleteness theorem (and perhaps even Wittgenstein's as well) seems a great deal more interesting.

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